

# THE $\mu$ -PERMANENT, A NEW GRAPH LABELING, AND A KNOWN INTEGER SEQUENCE

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ABSTRACT. Let  $A = (a_{ij})$  be an  $n$ -by- $n$  matrix. For any real number  $\mu$ , we define the polynomial

$$P_\mu(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \mu^{\ell(\sigma)},$$

as the  $\mu$ -permanent of  $A$ , where  $\ell(\sigma)$  is the number of inversions of the permutation  $\sigma$  in the symmetric group  $S_n$ . In this note, motivated by this notion, we discuss a new graph labeling for trees whose matrices satisfy certain  $\mu$ -permanental identities. We relate the number of labelings of a path with a known integer sequence. Several examples are provided.

## 1. INTRODUCTION

Given an  $n \times n$  matrix  $A = (a_{ij})$  and a real number  $\mu$ , we define the  $\mu$ -permanent of  $A$  as the polynomial

$$(1.1) \quad P_\mu(A) = \sum_{\sigma \in S_n} \left( \prod_{i=1}^n a_{i\sigma(i)} \right) \mu^{\ell(\sigma)},$$

where  $\ell(\sigma)$  is the number of inversions of the permutation  $\sigma$  in the symmetric group  $S_n$  of degree  $n$ , i.e., the number of interchanges of consecutive elements necessary to arrange  $\sigma$  in its natural order [13, p.1] or, equivalently,

$$\ell(\sigma) = \#\{(i, j) \in \{1, \dots, n\}^2 \mid i < j \text{ and } \sigma(i) > \sigma(j)\}.$$

For example, we have

$$P_\mu \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = a_{11}a_{22} + a_{12}^2\mu$$

and

$$P_\mu \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{11}a_{23}^2\mu + a_{12}^2a_{33}\mu + 2a_{12}a_{23}a_{13}\mu^2 + a_{13}^2a_{22}\mu^3.$$

The  $\mu$ -permanent of a square matrix is a natural extension of the determinant (setting  $\mu = -1$ ) and the permanent (setting  $\mu = 1$ ) which is in fact quite hard to compute [5, p.190]. In addition, making  $\mu = 0$ , we get the product of the main diagonal entries of the matrix.

This concept was introduced independently and almost simultaneously by different authors by a quarter of a century ago under different names in matrix theory and, surprisingly, in Grassmann algebras and quantum groups:  $q$ -permanent is just one of

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them [3–5, 9, 11, 12, 14, 15, 18–20]. Here we adopt one of the possible ways to call this matricial function (cf. also [6–8]).

Listing 1 presents a Mathematica [21] routine for computing the  $\mu$ -permanent of a square matrix.

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LISTING 1. Routine to compute the  $\mu$ -permanent

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inversionList[s_] := Module[{i, inverse = Ordering[s]},
  Table[Length[Select[Take[s, inverse[[i]]], (# > i)&]],
    {i, Length[s] - 1}]]

inversions[s_] := Apply[Plus, inversionList[s]]

permanent[A_, mu_] := Module[{n = Length[A]},
  Sum[Product[A[[i, s[[i]]]], {i, n}] * mu^inversions[s],
    {s, Permutations[Range[n]]}]]
```

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We believe that this code will be particularly useful for further developments on the properties of the  $\mu$ -permanent.

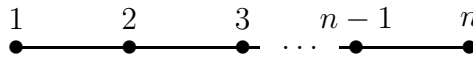
It is clear that in general, under similarity, the  $\mu$ -permanent does not keep the same value, i.e., the polynomial  $P_\mu(A)$  is not necessarily the same as  $P_\mu(BAB^{-1})$ , for  $B$  nonsingular. In particular, for permutation similarity. This means that interchanging rows and columns of the same indexes leads to possible different  $\mu$ -permanents. Since interchanging rows and columns does not change the underlying graph of the matrices involved, but the labeling of the vertices, we conclude that the  $\mu$ -permanent of a graph depends on its labeling.

After a first attempt to extend monotonic properties of the  $\mu$ -permanent of Jacobi positive definite matrices to more general acyclic matrices [8], it has recently been noticed a particular labeling for which the previous properties were indeed satisfied [6].

In this note we aim to discuss this new labeling for trees, counting them for paths. Incidentally, this process will lead to a new interpretation for a well-known integer sequence.

## 2. A NEW GRAPH LABELING

Given a symmetric matrix  $A$ , the graph of  $A$  is defined by the zero-nonzero off-main diagonal pattern of  $A$ . In general, the vertex labeling is not much discussed in matrix theory since most of the results involve the spectra of matrices, which do not change by such labelings. For example, the underlying graph of a tridiagonal matrix is a path with the vertices ordered successively  $1, 2, \dots, n$  and edges joining consecutive vertices  $i$  and  $i + 1$ :



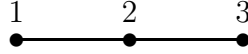
However, as we mentioned in the introduction, this is not the case for the  $\mu$ -permanent of a square matrix. For instance, we have

$$P_\mu \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{11}a_{23}^2\mu + a_{12}^2a_{33}\mu,$$

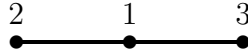
which is a polynomial of degree 1, and

$$P_\mu \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & 0 \\ a_{13} & 0 & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}^2a_{33}\mu + a_{13}^2a_{22}\mu^3,$$

which in turn has degree 3. The “graph” is the same but the labeling is not, i.e.,



and



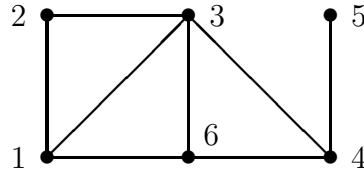
respectively.

In order to establish several general results for the  $\mu$ -permanent, recently in [6], the second author introduced the following labeling:

**Labeling 1.** *Given two disjoint edges  $ij$  and  $k\ell$ , say  $i < j$ ,  $k < \ell$ , and  $i < k$ , then one of the following conditions must be fulfilled:*

- (i)  $i < j < k < \ell$
- (ii)  $i < k < \ell < j$ .

To the best of our knowledge, this labeling is new and will be referred to in the remaining of this paper as a  $\mu$ -labeling. For example, we have the following:



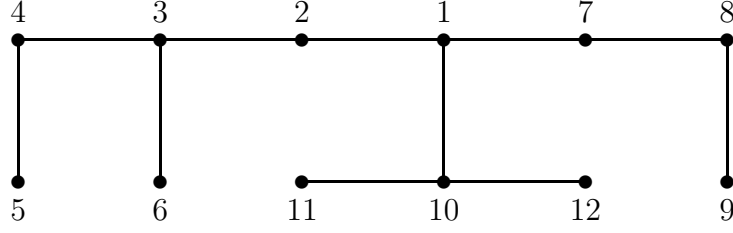
Interestingly, not all graphs allow such labeling: a complete graph of more than 3 vertices is just an example. However, any tree allows labelings satisfying the conditions described. In what follows, we present an algorithm to construct one of such labeling:

**Algorithm 1.** *Let us consider a tree with a given number of vertices.*

- Step 1. *Choose any vertex from the tree and label it with 1, which will be the root.*
- Step 2. *Take the largest path attached to vertex 1.*
- Step 3. *Label the vertices of this path by  $2, 3, \dots, k$ , where  $(i, i+1)$  is an edge, for  $i = 1, 2, \dots, k-1$ .*
- Step 4. *Choose the vertex with largest label in the previous with degree greater than two, say  $\ell$ .*
- Step 5. *Repeat step 2., replacing 1 by  $\ell$ .*
- Step 6. *Repeat step 3., labeling the vertices of the path by  $k+1, \dots, k'$ .*
- Step 7. *Once all vertices of degree more than two were considered, restart from 2., choosing the second largest path attached to the root and proceed until all vertices were considered.*

We remark that, if there is more than one path attached to the root of the same largest size we choose arbitrarily one of them. Clearly, this is not the only way to construct such labeling.

As a simple example of the algorithm, we have



Returning to the  $\mu$ -permanent, as a consequence, for any  $n \times n$  matrix  $A$  whose graph is a tree with the vertices labeled as described before, one always has

$$P_\mu(A) = a_{ii}P_\mu(A_i) + \sum_{i \sim j} |a_{ij}|^2 P_\mu(A_{ij}) \mu^{\ell(ij)},$$

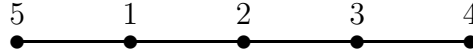
for any vertex  $i$ , or

$$\frac{d}{d\mu} P_\mu(A) = \sum_{i \sim j} \ell(ij) |a_{ij}|^2 P_\mu(A_{ij}) \mu^{\ell(ij)-1},$$

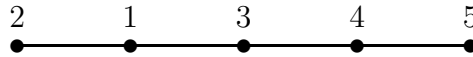
with  $i < j$ , (cf. [6–8]). Here,  $A_S$  is the matrix obtained from  $A$  replacing the rows and columns indexed by  $S$ , by zero, except the entries in the main diagonal, which are 1's.

### 3. COUNTING LABELINGS FOR PATHS

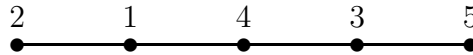
In this section we confine our study to paths. Our algorithm provides the following example for a path of 5 vertices:



The following labeling is also a possibility



but



is not.

A Mathematica routine that computes all the possible  $\mu$ -labelings for an order  $n$  path is given in Listing 2. For example, the distinct  $\mu$ -labelings for a path with for 4 vertices are

$$\begin{array}{cccc} \{1, 2, 3, 4\} & \{1, 2, 4, 3\} & \{1, 4, 2, 3\} & \{1, 4, 3, 2\} \\ \{2, 1, 3, 4\} & \{2, 3, 1, 4\} & \{2, 1, 4, 3\} & \{3, 2, 1, 4\} \end{array}.$$

Table 3.1 presents the number of distinct  $\mu$ -labelings for paths of order up to 11, which were computed with the same routines.

Remarkably, this exhaustive enumeration leads us exactly to the integer sequence A001792 of the The On-Line Encyclopedia of Integer Sequences [17]. This sequence has many different interpretations. Originally, we will find it in [10] (cf. also [1, Table 22.3]) in the absolute value of the coefficients of  $x^n$  for the Chebyshev polynomials of the first kind  $T_{n+2}$ . The most simple formula is perhaps  $(n+2)2^{n-1}$ , for each positive integer  $n$ . This sequence emerges also from the Bernoulli's triangle rows sums [2, 16]. Nonetheless,

LISTING 2. Mathematica routine to test and compute  $\mu$ -labelings

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```

qPermutations[n_] := Flatten[Table[Flatten[{i, #, j}] & /@
  Permutations[Complement[Range[n], {i, j}]],
  {i, n}, {j, n, i + 1, -1}], 2]

testPermutation[perm_] :=
  Module[{pair1, pair2, i, j, k, l},
    Catch[
      For[p1 = 1, p1 <= Length@perm - 3, ++p1,
        pair1 = Sort@perm[{{p1, p1 + 1}}];
        For[p2 = p1 + 2, p2 < Length@perm, ++p2,
          pair2 = Sort@perm[{{p2, p2 + 1}}];
          If[First@pair1 < First@pair2,
            {i, j} = pair1; {k, l} = pair2,
            {i, j} = pair2; {k, l} = pair1];
          If[i > k || (j > k && l > j), Throw[False]];
        ]
      ];
    Throw[True]
  ]

labelings[n_] := Select[qPermutations[n], testPermutation]

```

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TABLE 3.1. Total number of  $\mu$ -labelings

order	#labelings	order	#labelings
2	1	7	112
3	3	8	256
4	8	9	576
5	20	10	1280
6	48	11	2816

we can recent find a vast number of interesting interpretations for this sequence. Namely, it is the determinant of the square matrix with 3's on the diagonal and 1's elsewhere, or the absolute value of the determinant of the Toeplitz matrix with first row containing the first  $n$  integers [17].

Regarding other Mathematica routines, one can find several collected in [17] such as:

```

matrix[n_Integer /; n >= 1] := Table[Abs[p - q] + 1, q, n, p, n];
a[n_Integer /; n >= 1] := Abs[Det[matrix[n]]]

```

or

```

g[n_, m_, r_] := Binomial[n - 1, r - 1] Binomial[m + 1, r] r;

```

or

```

Table[1 + Sum[g[n, k - n, r], r, 1, k, n, 1, k - 1], k, 1, 29]

```

or

`LinearRecurrence[4, -4, 1, 3, 40]`

or even

`CoefficientList[Series[(1 - x) / (1 - 2 x)^2, x, 0, 40], x]`

One interesting open question is formally prove that this new labeling leads to the integer sequence A001792.

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